$$\theta = \begin{cases} -\alpha r^2 - \beta z, & \alpha, \beta = \text{const} > 0 \\ r \sum_{n=1}^{\infty} Z_1(\mu_n r) \exp(\mu_n z) \end{cases}$$

where Z_1 is a Bessel function of first order and first or second kind depending on the type of the complex quantity μ_n. Here the solution φ has a fairly complex form

$$\varphi = \text{const} \begin{cases} \exp(-\alpha r^2 - \beta z) \\ \exp\left[r \sum_{n=1}^{\infty} Z_1(\mu_n r) \exp(\mu_n z)\right] \end{cases}$$

The one-dimensional case of a cylindrical screw flow described by the exact solution of system (4.2)

$$\varphi = \text{const} \cdot \exp(-\alpha r^2), \quad \alpha = \text{const} > 0$$

which corresponds to the velocity field

$$v_\varphi = \varepsilon r^{-1}[1 - \exp(-\alpha r^2)], \quad v_z = 2\alpha r v_\varphi$$

lends itself to clearer interpretation. The solution is identical with the known solution for a time-limited state of a rectilinear vortex stretching in axial flow (Burgers vortex). The solution was used by a number of authors as a heuristic model of a twisted external flow, while studying the mechanism of vortex collapse /7/. The magnitude of the constant α can be estimated from the condition $v_\varphi, v_z = O(1)$: $\alpha \sim \varepsilon^{-1}$ corresponding to the experimental results obtained by Garg /7/.

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ON STRONG TRANSITIONS BETWEEN STRUCTURES OF DIFFERING SYMMETRY ACCOMPANYING WEAKLY SUPERCRITICAL CONVECTION*

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A complete classification of the phase space of the dynamical system which describes the motion of a liquid when there is weakly supercritical convection is carried out within the framework of a six-mode Galerkin approximation. It is shown that all the phase trajectories are attracted to the corresponding stationary states. The domains of attraction to each of these states are found. The minimum value of a perturbation, which converts a weakly stable solution of one symmetry into a stable solution of another symmetry when the parameters of the problem are close to their bifurcation values, is estimated.

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It is known that, in certain cases when there is weakly super-critical convection, two types of stationary spatially periodic flows may arise which, in typical situations, are cellular hexagonal structures and structures in the form of two dimensional axles. When this is so, both types of structures are found to be stable with respect to small perturbations in a certain domain of the values of the parameters of the problem so that strong transitions between them become possible under the action of perturbations of finite amplitude /1-3/. In such cases the question as to the minimum amplitude of a perturbation which converts a structure of one symmetry into a structure of another symmetry, and questions relating to this regarding a complete classification of the possible asymptotic states of the dynamical system under consideration and the domains of attraction of different initial conditions to these asymptotic states are of interest. This paper is concerned with investigating these questions.

Let us start out from the system of equations obtained in /1/ in a finite mode (Galerkin) approximation, a rigorous proof of which is given in /4/. In this approximation the components of the velocity vector for a convective flow and the perturbation of the temperature profile can be represented in the form

$$\sum_{j=1}^{3}\{A_j\cos(\mathbf{q}_j\mathbf{r})+B_j\sin(\mathbf{q}_j\mathbf{r})\}f(z) \tag{1}$$

Here, $\mathbf{r}$ is a two dimensional radius vector lying in the horizontal plane (it is assumed that the convection layer has an infinite extension in this plane), the z-axis is directed along the gravitational force and the two dimensional vectors $\mathbf{q}_i$ satisfy the relationship $|\mathbf{q}_1|=|\mathbf{q}_2|=|\mathbf{q}_3|;\ \mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3=0$. In this case the problem reduces to determining the quantities $A_j(t)$ and $B_j(t)$ for which the closed system

$$Z_1^{\cdot}=\varepsilon Z_1-|Z_1|^2Z_1-2\mu(|Z_2|^2+|Z_3|^2)Z_1-2\alpha Z_2^*Z_3^* \tag{2}$$

follows from the equations of hydrodynamics in the third order of perturbation theory (a further two equations are obtained from (2) by cyclic permutation of the indices 1, 2 and 3) The dot denotes a derivative with respect to time t; $Z_j\equiv A_j+iB_j$; ε is the bifurcation parameter of the problem, which is proportional to $(R-R_0)/R_c$, where R is a dimensionless quantity (the Rayleigh number) which is proportional to the difference in the temperatures between the lower and upper surfaces of the convection layer and R_c is that value of R at which the resting liquid loses stability with respect to perturbations which may be as small as may be desired and lead to the occurrence of convective motion.

Eq.(2) is written in a suitable manner using dimensionless variables whereupon the numerical values of the real positive coefficients μ and α depend on the actual formulation of the convection problem (the form of the boundary conditions, etc.). In typical cases, these quantities satisfy the relationships $\mu\sim 1$ and $\alpha\ll 1$ and the whole of the approximation being considered is applicable subject to the condition that $|\varepsilon|\ll 1$. Moreover, in the majority of convection problems, μ satisfies the inequality $\mu>{}^1/_2$ and this is subsequently assumed.

In proceeding to an analysis of system (2) we note that it belongs to the so-called systems of gradient form, that is, it can be written as

$$Z_j^{\cdot}=-\partial H/\partial Z_j^* \tag{3}$$

where the pseudo-Hamiltonian H (the Lyapunov function) is defined using the right-hand side of (2). It follows from (3) that

$$H^{\cdot}=-2\sum_j|\partial H/\partial Z_j|^2\equiv-2\sum_j|Z_j^{\cdot}|^2\leqslant 0 \tag{4}$$

Hence, all of the phase trajectories of system (2) must tend to certain stationary points which are local extrema of the quantity H (for large $|Z|$, the latter increases monotonically as $|Z|$ increases which, when accounts is taken of (4), does not permit the phase trajectories of system (2) to go "to infinity").

Hence, in order to provide a complete description of the qualitative properties of system (2), it is sufficient to find all the singular points and to determine the form of the separatrix surface which define the boundaries of the domains of attraction of the various stable singular points.

Next, we note that the initial hydrodynamic problem possesses obvious symmetry with respect to a translation in the horizontal plane by an arbitrary constant vector $\mathbf{a}$. The rotation:

$$Z_j\to Z_j\exp(i\psi_j) \tag{5}$$

corresponds to the transformation $\mathbf{r} \to \mathbf{r} + \mathbf{a}$ in Z-space, where the phases ψ_j are connected by the relationship $\psi_1 + \psi_2 + \psi_3 = 0$. In order to eliminate degeneracy due to the existence of this symmetry it is convenient to transform from the variables Z to the new independent variables I which are invariant with respect to the transformation (5). There are just four independent invariants for which, for example, the following may be taken:

$$I_{1,2,3} = |Z_{1,2,3}|^2, \; I_4 = \mathrm{Im}\,(Z_1 Z_2 Z_3) \tag{6}$$

We may choose $\varphi_{1,2} = \arg Z_{1,2}$ as the two remaining variables which extend the four-dimensional phase space (6) up to the six dimensional space of system (2). The equations for I and φ, which are obtained from (2), have the form

$$I_1^{\cdot} = 2\,\{\varepsilon I_1 - I_1^2 - 2\mu\,(I_2 + I_3) I_1 - 2\alpha I_5\} \tag{7}$$

$$I_4^{\cdot} = \{3\varepsilon - (4\mu + 1)(I_1 + I_2 + I_3)\} I_4 \tag{8}$$

$$\varphi_j^{\cdot} = 2\alpha I_4 / I_j \;\; (I_5 \equiv \mathrm{Re}\,(Z_1 Z_2 Z_3)) \tag{9}$$

(the equations for I_2 and I_3 follow from (7) on cyclic permutation of the indices 1, 2 and 3)

By taking account of the fact that the invariant I_5 is connected with $I_{1,2,3,4}$ by the relationship $I_5^2 = I_1 I_2 I_3 - I_4^2$, we obtain that the four Eqs.(7) and (8) for the invariants form a closed subsystem. This means that the six-dimensional phase space of system (2) is stratified into, generally speaking, two-dimensional manifolds (the orbits of the transformation (5)), which are defined by the equations $I_j = \mathrm{const}$.

Next, we note that, in the four-dimensional phase space of system (7), (8), there is a three-dimensional invariant subspace, i.e. the hyperplane $I_4 = 0$. It follows from (8) that the phase trajectories of system (7), (8) are repelled from this hyperplane in the domain

$$I_1 + I_2 + I_3 < 3\varepsilon/(4\mu + 1)$$

and attracted to it outside of this domain. The above-mentioned hyperplane $I_4 = 0$ and the surface $I_1 + I_2 + I_3 = 3\varepsilon/(4\mu + 1)$, which, in six-dimensional Z-space, corresponds to a sphere with its centre at the origin of coordinates, are the zero-isoclines of Eq.(8). It is easy to show by means of simple calculations that there is no singular point of Eqs.(2) on the surface of this sphere.

This means that all the singular points of system (7), (8) lie on the hyperplane $I_4 = 0$. It follows from this and what has been said previously that, at sufficiently large values of t, all the phase trajectories of system (7), (8) which do not belong to the above-mentioned hyperplane must approach it asymptotically. It is for this reason that all the information concerning the asymptotic behaviour of system (7) and (8) (and, consequently, also concerning the initial system (2)) can be extracted from an analysis of its phase protrait in the invariant subspace $I_4 = 0$. When this is done, by using the transformation (5), all equivalent trajectories can be reduced to those for which $\mathrm{Im} Z_1 = \mathrm{Im} Z_2 = \mathrm{Im} Z_3 = 0$, i.e. $Z_j \equiv A_j$.

It is seen that the dynamical system which is obtained in this case has six invariant planes $A_j = \pm A_l$ $(j \neq l)$ which play an important role in the analysis of its phase portrait.

Of course, corresponding invariant subspaces also exist in Eqs.(7)-(9) and in the initial system (2). The existence of these invariant manifolds (and also of the manifold $I_4 = 0$) is not accidentally associated with the occurrence of a certain symmetry in the problems. Thus, apart from the translational symmetry which has been noted above, the problem is symmetric with respect to a complex conjugate transformation (whence, the invariance of the manifold $I_4 = 0$) and with respect to permutations associated with the renumbering of the basis vectors $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3$ (the invariance of the planes $A_j = \pm A_l$). For the above-mentioned reasons the invariance of these manifolds is not destroyed when account is taken in (2) of the corrections associated with the higher orders of perturbation theory which lead to the appearance of terms $\sim Z^n$, where $n \geqslant 4$ on the right-hand side of (2).

In particular, it follows from this that the lines of intersection of the invariant planes, that is, the lines defined by the equations $\pm A_1 = \pm A_2 = \pm A_3$, where each sign is selected independently, are point phase trajectories of this system.

Let us now consider the subsystem which is obtained on reducing this dynamical system on one of the above-mentioned invariant planes. To be specific, let us select the plane $A_1 = A_3$ (Figs.1-3). The corresponding dynamical equations have the form

$$\begin{aligned} A_1^{\cdot} &= \varepsilon A_1 - A_1^3 - 2\mu\,(A_1^2 + A_2^2)\,A_1 - 2\alpha A_1 A_2 \\ A_2^{\cdot} &= \varepsilon A_2 - A_2^3 - 4\mu A_1^2 A_2 - 2\alpha A_1^2 \end{aligned} \tag{10}$$

(on reduction onto the plane $A_1 = -A_3$, a system is obtained which differs from (10) in the sign in front of the coefficient α).

Apart from the trivial singular point $A_1 = A_2 = 0$, system (10) has the following singular points /4/:

$$A_1 = 0, \quad A_2^2 = \varepsilon \tag{11}$$

$$\pm A_1 = A_2 = -\frac{\alpha \pm \sqrt{\alpha^2 + (4\mu+1)\varepsilon}}{4\mu+1} \tag{12}$$

$$A_1^2 = \frac{\varepsilon - \varepsilon_R}{2\mu+1}, \quad A_2 = -\sqrt{\varepsilon_R} \quad \left(\varepsilon_R \equiv \frac{\alpha^2}{(\mu - 1/2)^2}\right) \tag{13}$$

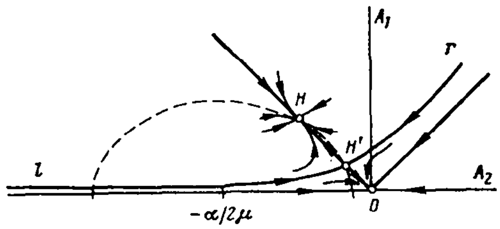

Fig.1

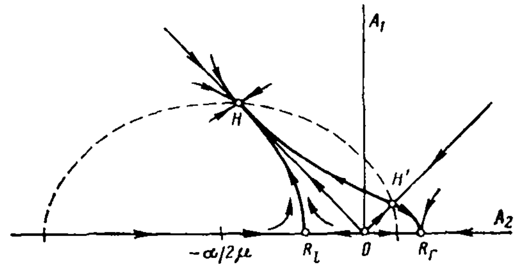

Fig.2

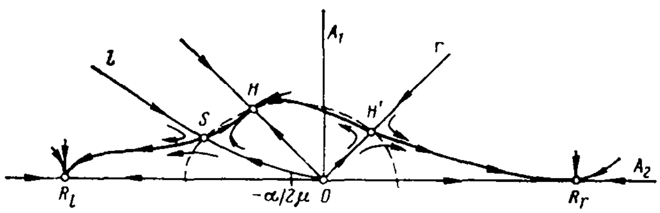

Fig.3

(the signs in front of A_1 and the square root sign in (12) are chosen independently). It can be seen from (1) that the points (11) correspond to shafts while the points (12) correspond to hexagons. The points (13) correspond to a new stationary state, that is, to warped hexagons.

By taking account of the symmetry of the problem, it is readily seen that, in the three dimensional phase space (A_1, A_2, A_3) there are six singular points of type (11), each of which simultaneously belongs to two invariant planes, eight singular points of type (12), each of which simultaneously belongs to three invariant planes and twelve singular points of type (13) through each of which just a single invariant plane passes. Allowing for the trivial singular point at the origin of coordinates the total number of singular points is equal to $6 + 8 + 12 + 1 = 27$.

On the other hand, the system of three cubic equations which is obtained from (2) when $Z_1^{\cdot} = Z_2^{\cdot} = Z_3^{\cdot} = 0$ has not more than 3^3 real solutions, that is, all the non-trivial singular points of the system lying in the phase space (A_1, A_2, A_3) are taken up by the points (11)-(13).

A subdivision of this three-dimensional phase space into the domains of attraction of the different stable stationary states can be synthesized from the two-dimensional phase portraits of a reduced system of the type of (10), to the study of which we shall now pass. In doing this we shall denote the singular points (11) by $R_{l,r}$, where the index l refers to the left point $(A_2 = -\sqrt{\varepsilon})$ and r refers to the right point $(A_2 = \sqrt{\varepsilon})$. The singular points (12) which correspond to the larger values of $|A|$ (the upper sign) are denoted by the letter H and those corresponding to the smaller values of $|A|$ (the lower sign) are denoted by H' while points of the type of (13) are denoted by the letter S.

In describing the bifurcations we shall assume that the magnitude of α is fixed and study the change in the stability of the individual solutions as the supercriticality parameter ε /4/ is increased. Furthermore, since the phase picture in Figs.(1)-(3) is symmetrical about the A_2 axis, we shall confine ourselves to the description of that part of it which lies in the upper half plane (the phase picture in the plane $A_1 = -A_3$ is obtained from the analogous picture on the $A_1 = A_3$ plane upon its mirror reflection with respect to the A_1 axis. This removes the apparent asymmetry of the problem associated with the different form of the phase portrait of system (10) in the right-hand and left-hand half planes of Figs.1-3).

The analysis leads to the following results.

1°. When $\varepsilon < \varepsilon_{min} \equiv -\alpha^2/(4\mu + 1)$, not one of the singular points (11)-(13) exists. The sole singular point of system (10) is the trivial singular point $A_1 = A_3 = 0$ (the stable node).

2°. When $\varepsilon = \varepsilon_{min}$ a complex singular point $H(H')$, a saddle node is created in the phase plane (A_1, A_3) which, when $\varepsilon > \varepsilon_{min}$, is split into a stable node H which corresponds to stable hexagons and a saddle H'. Moreover, at the "moment" $\varepsilon = \varepsilon_{min}$, the finite domain of attraction to the point $H(H')$ arises from the discontinuity. The latter is explained by the fact that, when $0 < \varepsilon_{min} - \varepsilon \ll |\varepsilon_{min}|$, the point $H(H')$ exists in a virtual form: the trajectories are attracted to the site where it must appear and then pass out onto the point O along a direction close to the unstable separatrix to be $H'O$ (see Fig.1, where a phase portrait of system (10) is given when $\varepsilon_{min} < \varepsilon < 0$ (the ellipse $(2\mu + 1) A_1^2 + 2\mu (A_3 + \alpha/2\mu)^2 = \varepsilon + \alpha^2/2\mu$ which is one of the branches of the zero-isocline $\dot{A_1} = 0$ is represented by the broken lines in Figs.1-3) and the second branch of the zero-isocline coincides with the A_3 axis).

In the domain $0 < \varepsilon - \varepsilon_{min} \ll |\varepsilon_{min}|$ the minimum amplitude δA_{min} of a perturbation which destroys the weakly stable hexagonal structure, that is, which transfers the system from the stable point H to beyond the separatrix $lH'r$ (Fig.1), is estimated by the expression

$$\delta A_{min} \sim \sqrt{\varepsilon - \varepsilon_{min}} \qquad (14)$$

It can be shown that, in the phase space (A_1, A_2, A_3), the shortest distance form the point H to the separatrix of the surface is the distance between the points H and H' so that the estimate (14) may be replaced by the rigorous expression:

$$\delta A_{min} = 2\sqrt{\frac{3}{4\mu + 1}(\varepsilon - \varepsilon_{min})}$$

In the general case the estimation of δA_{min} is based on the fact that, when there are bifurcations of the type under consideration, the direction along which the separatrix surface approaches closest of all to the weakly stable singular point is close to the direction of the eigenvector associated with the greatest characteristic index of this point (that is the smallest in absolute magnitude). We note that, at the moment of bifurcation, the corresponding characteristic index must vanish and, hence, such vectors cannot have components which are transverse to the hyperplane $I_4 = 0$. It follows from this that perturbations with $B_j \neq 0$ which transform the system from a weakly stable singular point through the separatrix surface always have an amplitude which is larger than the analogous perturbations with $B_j \equiv 0$. What has been said clarifies why, in the estimation of δA_{min}, it is possible to confine oneself within the framework of the three-dimensional phase space (A_1, A_2, A_3).

3°. When ε is increased further, the saddle H' passes through the point O at the "moment" $\varepsilon = 0$, converting it into an unstable node. When $\varepsilon > 0$, the point H' passes into the right-hand plane and two new singular points (11) are generated from the point O. These two new points are R_l and R_r which, in the three dimensional phase space (A_1, A_2, A_3) possess a saddle instability up to the "moment" $\varepsilon = \varepsilon_R$ (Fig.2).

4°. When $\varepsilon = \varepsilon_R$, two saddles S (13) originate from the point R_l which, when, $\varepsilon > \varepsilon_R$, pass out, one into the upper and one into the lower half plane of the plane $A_1 = A_3$ (Fig.3). At the same time the point R_l becomes a stable node, corresponding to stable shafts, the domain of attraction to which in the plane $A_1 = A_3$ is restricted to the stable separatrix of the saddle S which is split off from the semi-axis $A_3 < 0$ when $\varepsilon = \varepsilon_R$. Similar changes occur in the phase portrait in the neighbourhood of the point R_r but the saddles which are generated from it when $\varepsilon = \varepsilon_R$ lie in the $A_1 = -A_3$ plane.

The value of δA_{min} in the domain $0 < \varepsilon - \varepsilon_R \ll \varepsilon_R$ is estimated using a relationship which is analogous to (14) after replacing ε_{min} by ε_R.

5°. On further increasing ε the point S approaches the point H and, when $\varepsilon = \varepsilon_{max} \equiv 2(\mu + 1)\varepsilon_R$, passes through this point, imparting to it a saddle instability. Moreover, at the moment of bifurcation, that is, when $\varepsilon = \varepsilon_{max}$, the two characteristic indices of the points S in the three-dimensional phase space (A_1, A_2, A_3) simultaneously change sign. For instance, in the case of the point S, shown in Fig.3, the unstable characteristic direction lying, when $\varepsilon < \varepsilon_{max}$ in the invariant plane $A_1 = A_3$ becomes stable when $\varepsilon > \varepsilon_{max}$ while the characteristic direction which does not belong to this plane and is stable when $\varepsilon < \varepsilon_{max}$ is made unstable when $\varepsilon > \varepsilon_{max}$.

Hence, when $\varepsilon > \varepsilon_{max}$ the points R, corresponding to shafts, will be the only stable singular points of the dynamical system under consideration.

The separatrix surfaces which, when $\varepsilon < \varepsilon_{max}$, bound the domains of attraction of the points H in the space (A_1, A_2, A_3) have a third-order axis of symmetry OH which is the line of intersection of the three planes of symmetry. In the case of the point H shown in Fig.3, these planes are $A_1 = -A_2$; $A_1 = A_3$ and $A_2 = -A_3$. A cross-section of the separatrix surface corresponding to this point in a plane which is orthogonal to the straight line OH is shown in Fig.4 (the cross-section of the domain of attraction to the point H is hatched in).

As $\varepsilon \to \varepsilon_{max}$ $(\varepsilon < \varepsilon_{max})$ these separatrix surfaces approximate to surfaces of symmetry and, when $\varepsilon = \varepsilon_{max}$, the domains of attraction to the points H collapse into two dimensional manifolds, each of which consists of three sectors lying in the corresponding symmetry planes. In the case of the point H under consideration (Fig.3) in the plane $A_1 = A_3$ such a sector is bounded by the bisectors of the first and fourth coordinate angle.

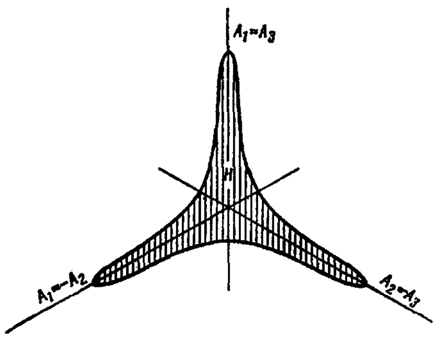

Fig.4

The estimate

$$\delta A_{min} \sim \varepsilon_{max} - \varepsilon \tag{15}$$

holds for the quantity δA_{min} in the domain $0 < \varepsilon_{max} - \varepsilon \ll \varepsilon_{max}$.

The difference between (15) and the square root law (14) is due to the fact that, when $\varepsilon = \varepsilon_{max}$, neither points S or points H are either generated or disappear but only pass through one another. It is for this reason that, in the law describing their displacement in phase space as ε is varied, the point $\varepsilon = \varepsilon_{max}$ is regular.

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